# CHARACTER TABLES OF ASSOCIATION SCHEMES BASED ON ATTENUATED SPACES

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Abstract. The set of subspaces of a given dimension in an attenuated space has a structure of a symmetric association scheme and this association scheme is called an association scheme based on an attenuated space. Association schemes based on attenuated spaces are generalizations of Grassmann schemes and bilinear forms schemes, and also $q$-analogues of non-binary Johnson schemes. Wang, Guo and Li computed the intersection numbers of association schemes based on attenuated spaces. The aim of this paper is to compute character tables of association schemes based on attenuated spaces using the method of Tarnanen, Aaltonen and Goethals. Moreover, we also prove that association schemes based on attenuated spaces include as a special case the $m$-flat association scheme, which is defined on the set of cosets of subspaces of a constant dimension in a vector space over a finite field.

## 1. Introduction

Let $\mathbb{F}_q$ be the finite field of size $q$ and $\mathbb{F}_q^N$ denotes the vector space of $N$-tuples over $\mathbb{F}_q$. For a positive integer $n$ and a non-negative integer $l$, we fix an $l$-dimensional subspace $\mathfrak{e}$ of $\mathbb{F}_q^{n+l}$. The corresponding *attenuated space* associated with $\mathbb{F}_q^{n+l}$ and $\mathfrak{e}$ is the collection of all subspaces of $\mathbb{F}_q^{n+l}$ intersecting trivially with $\mathfrak{e}$. For non-negative integers $m$ and $k$, an $m$-dimensional subspace $\mathfrak{p}$ of $\mathbb{F}_q^{n+l}$ is called a subspace of *type* $(m,k)$ with respect to $\mathfrak{e}$ if $\dim \mathfrak{p} \cap \mathfrak{e} = k$, and especially a subspace $\mathfrak{p}$ of type $(m,0)$ is an element of the attenuated space associated with $\mathbb{F}_q^{n+l}$ and $\mathfrak{e}$. Denote the set of all subspaces of type $(m,k)$ in $\mathbb{F}_q^{n+l}$ by $\mathcal{M}_q(m,k;n+l,n)$. The cardinality of $\mathcal{M}_q(m,k;n+l,n)$ is

$$q^{(m-k)(l-k)} \begin{bmatrix} n \\ m-k \end{bmatrix}_q \begin{bmatrix} l \\ k \end{bmatrix}_q,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i}$, i.e., the *Gaussian coefficient*. The subscript $q$ will be omitted when there is no possibility of confusion.

In 2009, Wang, Guo and Li proved that $\mathcal{M}_q(m,0;n+l,n)$ has a structure of a symmetric association scheme $\mathfrak{X}(\mathcal{M}_q(m,0;n+l,n))$ and they computed intersection numbers of $\mathfrak{X}(\mathcal{M}_q(m,0;n+l,n))$ [13]. In their paper, the relation $R_{(i,j)}$ on $\mathcal{M}_q(m,0;n+l,n)$ is defined to be the set of pairs $(\mathfrak{p},\mathfrak{q})$ satisfying

$$\dim((\mathfrak{p}+\mathfrak{e})/\mathfrak{e} \cap (\mathfrak{q}+\mathfrak{e})/\mathfrak{e}) = m-i \text{ and } \dim \mathfrak{p} \cap \mathfrak{q} = (m-i)-j,$$

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for $(i,j) \in K = \{(i,j) \in {\mathbb{Z}_{\geq 0}}^2 \mid i \leq m \wedge (n-m),\ j \leq (m-i) \wedge l\}$, where for integers $a$ and $b$, the value $a \wedge b$ denotes $\min\{a,b\}$ for short. Then $(\mathcal{M}_q(m,0;n+l,n), \{R_{(i,j)}\}_{(i,j)\in K})$ is a symmetric association scheme and called an *association scheme based on an attenuated space*. In this paper, we denote it by $\mathfrak{X}(\mathcal{M}_q(m,0;n+l,n))$.

The association scheme $\mathfrak{X}(\mathcal{M}_q(m,0;n+l,n))$ is a common generalization of the Grassmann scheme $J_q(n,m)$ and the bilinear forms scheme $H_q(n,l)$. In fact, if $l=0$, then the association scheme $\mathfrak{X}(\mathcal{M}_q(m,0;n,n))$ is isomorphic to the Grassmann scheme $J_q(n,m)$ and if $m=n$, then the association scheme $\mathfrak{X}(\mathcal{M}_q(n,0;n+l,n))$ is isomorphic to the bilinear forms scheme $H_q(n,l)$. Moreover the association scheme $\mathfrak{X}(\mathcal{M}_q(m,0;n+l,n))$ is also a $q$-analogue of the non-binary Johnson scheme (cf. [12]).

The aim of this paper is to determine the character tables of association schemes based on attenuated spaces. To determine the character tables, we use the method of Tarnanen, Aaltonen and Goethals [12].

Determining the character table of an association scheme corresponds to determining the spherical functions of a compact symmetric space. In [3], Bannai and Ito referred to an analogy between compact symmetric spaces of rank one and a family of association schemes which are called ($P$ and $Q$)-polynomial association schemes. In fact, zonal spherical functions of compact symmetric spaces of rank one and eigenmatrices of ($P$ and $Q$)-polynomial association schemes are described by certain orthogonal polynomials. Moreover Bannai [2] had the assurance that there exists an analogy between general compact symmetric spaces and most of commutative association schemes, and he observed relations between spherical functions of some compact symmetric spaces and character tables of some commutative association schemes. In order to study relations between compact symmetric spaces and commutative association schemes, it is useful to know many examples of character tables of commutative association schemes.

In Section 2, the main result of this paper will be described after giving some definitions and basic facts about association schemes. In Section 3, we prove Lemma 2.2, which is a key lemma of the proof of the main result. In Section 4, a relation between association schemes based on attenuated spaces and $m$-flat association schemes will be found. Namely, we prove that association schemes based on attenuated spaces include as a special case the $m$-flat association scheme, which is defined on the set of cosets of subspaces of a constant dimension in a vector space over a finite field. Finally in Appendix A we describe some useful formulas about the number of subspaces of a vector space over a finite field and equations related to the $q$-Gaussian coefficient, the generalized Eberlein polynomials and the generalized Krawtchouk polynomials, which are used in this paper.

## 2. Character tables of association schemes based on attenuated spaces

We begin with a review of basic definitions concerning association schemes. The reader is referred to Bannai-Ito [3] for the background material.

A *symmetric association scheme* $\mathfrak{X} = (X, \{R_i\}_{0\leq i\leq d})$ consists of a finite set $X$ and a set $\{R_i\}_{0\leq i\leq d}$ of binary relations on $X$ satisfying:

(1) $R_0 = \{(x,x) \mid x \in X\}$;
(2) $\{R_i\}_{0\leq i\leq d}$ is a partition of $X \times X$;

(3) ${}^{t}R_i = R_i$ for each $i \in \{0,1,\dots,d\}$, where ${}^{t}R_i = \{(y,x) \mid (x,y) \in R_i\}$;
(4) the numbers $|\{z \in X \mid (x,z) \in R_i \text{ and } (z,y) \in R_j\}|$ are constant whenever $(x,y) \in R_k$, for each $i,j,k \in \{0,1,\dots,d\}$.

Note that the numbers $|\{z \in X \mid (x,z) \in R_i \text{ and } (z,y) \in R_j\}|$ are called the *intersection numbers* and denoted by $p^k_{i,j}$. Let $M_X(\mathbb{C})$ denote the algebra of matrices over the complex field $\mathbb{C}$ with rows and columns indexed by $X$. The $i$-th *adjacency matrix* $A_i$ in $M_X(\mathbb{C})$ of $\mathfrak{X}$ is defined by

$$A_i(x,y) = \begin{cases} 1 & \text{if } (x,y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

The vector space $\mathfrak{A} = \langle A_0, A_1, \dots, A_d \rangle_{\mathbb{C}}$ spanned by $A_i$'s $(i = 0,1,\dots,d)$ forms a commutative algebra and called the *Bose-Mesner algebra* of $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$. It is well known that $\mathfrak{A}$ is semi-simple, hence $\mathfrak{A}$ has a basis consisting of the primitive idempotents $E_0 = 1/|X|J, E_1, \dots, E_d$, i.e., $E_iE_j = \delta_{i,j}E_i$, $\sum_{i=0}^{d} E_i = I$, where $J$ is the all-one matrix and the notation $\delta_{i,j}$ stands for the value 1 if $i = j$, 0 otherwise. The *first eigenmatrix* $P = (P_i(j))_{0\le j,i\le d}$ and the *second eigenmatrix* $Q = (Q_i(j))_{0\le j,i\le d}$ of $\mathfrak{X}$ are defined by

$$A_i = \sum_{j=0}^{d} P_i(j)E_j \text{ and } E_i = \frac{1}{|X|}\sum_{j=0}^{d} Q_i(j)A_j,$$

respectively. In particular, the first eigenmatrix $P$ is also called the *character table* of $\mathfrak{X}$. Note that $P$ and $Q$ satisfy $PQ = QP = |X|I$, and it is well known that

$$\frac{Q_j(i)}{m_j} = \frac{P_i(j)}{v_i}, \tag{2.1}$$

where $v_i = p^0_{i,i}$ is called the $i$-th *valency* and $m_j = \mathrm{Tr}\,(E_j)$ is called the $j$-th *multiplicity*.

Next we give two examples of association schemes. The eigenmatrices of association schemes based on attenuated spaces will be given using the entries of the eigenmatrices of these association schemes. Let $V$ and $E$ be $n$-dimensional and $l$-dimensional vector spaces over $\mathbb{F}_q$, respectively, and let $L(V,E)$ denote the set of all linear maps from $V$ to $E$. For a subspace $\mathfrak{p}$ of $V$, the set of all $r$-dimensional subspaces of $\mathfrak{p}$ is denoted by $\left[\begin{smallmatrix}\mathfrak{p}\\ r\end{smallmatrix}\right]$.

The set $\left[\begin{smallmatrix}V\\ m\end{smallmatrix}\right]$ together with the nonempty relations

$$R_i = \{(\mathfrak{x},\mathfrak{y}) \in \left[\begin{smallmatrix}V\\ m\end{smallmatrix}\right]^2 \mid \dim \mathfrak{x}\cap\mathfrak{y} = m - i\}$$

is an $m\wedge(n-m)$-class symmetric association scheme called the *Grassmann scheme* $J_q(n,m)$. The first eigenmatrix $P^G = (P^G_k(x))_{0\le x,k\le m\wedge(n-m)}$ of the Grassmann scheme $J_q(n,m)$ is given by the *generalized Eberlein polynomials* $E_k(n,m;q;x)$ [7], namely

$$\begin{aligned} P^G_k(x) &= E_k(n,m;q;x) \\ &= \sum_{j=0}^{k} (-1)^{k-j} q^{jx+\binom{k-j}{2}} \begin{bmatrix} m-j \\ m-k \end{bmatrix} \begin{bmatrix} m-x \\ j \end{bmatrix} \begin{bmatrix} n-m+j-x \\ j \end{bmatrix}. \end{aligned}$$

Furthermore, since the valencies $v_i$ and the multiplicities $m_j$ of the Grassmann scheme $J_q(n,m)$ are given as $v_i = q^{i^2}\left[\begin{smallmatrix}n-m\\ i\end{smallmatrix}\right]\left[\begin{smallmatrix}m\\ i\end{smallmatrix}\right]$ and $m_j = \left[\begin{smallmatrix}n\\ j\end{smallmatrix}\right] - \left[\begin{smallmatrix}n\\ j-1\end{smallmatrix}\right]$, respectively

(cf. [4, p. 269]), by (2.1) we obtain the second eigenmatrix $Q^G = (Q_k^G(x))_{0\le x,k\le m\wedge(n-m)}$ of the Grassmann scheme $J_q(n,m)$ as follows;

$$\begin{aligned} & Q_k^G(x) \\ = \; & \frac{\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix}}{q^{x^2}\begin{bmatrix} n-m \\ x \end{bmatrix}\begin{bmatrix} m \\ x \end{bmatrix}} \sum_{j=0}^{x} (-1)^{x-j} q^{jk+\binom{x-j}{2}} \begin{bmatrix} m-j \\ m-x \end{bmatrix} \begin{bmatrix} m-k \\ j \end{bmatrix} \begin{bmatrix} n-m+j-k \\ j \end{bmatrix}. \end{aligned}$$

We denote this value by $Q_k(n,m;q;x)$.

The set $L(V,E)$ together with the nonempty relations

$$R_i = \{(f,g) \in L(V,E)^2 \mid \operatorname{rank}(f-g) = i\}$$

is an $n \wedge l$-class symmetric association scheme called the *bilinear forms scheme* $H_q(n,l)$ [8]. The first eigenmatrix $P^B = (P_k^B(x))_{0\le x,k\le n\wedge l}$ of the bilinear forms scheme $H_q(n,l)$ is given by the *generalized Krawtchouk polynomials* $K_k(n,l;q;x)$ [7], namely

$$\begin{aligned} P_k^B(x) &= K_k(n,l;q;x) \\ &= \sum_{j=0}^{k} (-1)^{k-j} q^{jl+\binom{k-j}{2}} \begin{bmatrix} n-j \\ n-k \end{bmatrix} \begin{bmatrix} n-x \\ j \end{bmatrix}. \end{aligned}$$

Note that bilinear forms schemes are self-dual, i.e., $P^B = Q^B$, where $Q^B$ is the second eigenmatrix of $H_q(n,l)$ (see [8]). Hence, by $P^BQ^B = |L(V,E)|I$, we obtain

$$\sum_{k=0}^{n\wedge l} K_k(n,l;q;i) K_j(n,l;q;k) = q^{nl}\delta_{i,j}. \tag{2.2}$$

In order to calculate the character table of the association scheme $\mathfrak{X}(\mathcal{M}_q(m,0;n+l,n))$, we deal with another realization of $\mathfrak{X}(\mathcal{M}_q(m,0;n+l,n))$. Let $\mathcal{X}$ be the set of all pairs of a subspace $\mathfrak{x}$ in $V$ and a linear map $\xi$ in $L(\mathfrak{x},E)$ and let $\mathcal{X}_m = \{(\mathfrak{x},\xi) \in \mathcal{X} \mid \mathfrak{x} \in \begin{bmatrix} V \\ m \end{bmatrix}\}$. The set $\mathcal{X}_m$ has cardinality $q^{ml}\begin{bmatrix} n \\ m \end{bmatrix}$. Then there is the following one-to-one correspondence between $\mathcal{X}_m$ and $\mathcal{M}_q(m,0;n+l,n)$: we regard $V \oplus E$ as $\mathbb{F}_q^{n+l}$ and also regard $\{0\} \oplus E$ as $\mathfrak{e}$. For $(\mathfrak{x},\xi)$ in $\mathcal{X}_m$, we set $\mathfrak{x}_\xi = \{(x,\xi(x)) \mid x \in \mathfrak{x}\}$ in $\mathbb{F}_q^{n+l}$. Then we immediately check that $\mathfrak{x}_\xi$ is type $(m,0)$ with respect to $\mathfrak{e}$, i.e., $\mathfrak{x}_\xi \in \mathcal{M}_q(m,0;n+l,n)$, and the map $(\mathfrak{x},\xi) \mapsto \mathfrak{x}_\xi$ from $\mathcal{X}_m$ to $\mathcal{M}_q(m,0;n+l,n)$ is injective. On the other hand, we have $|\mathcal{X}_m| = |\mathcal{M}_q(m,0;n+l,n)| = q^{ml}\begin{bmatrix} n \\ m \end{bmatrix}$. This means that the map from $\mathcal{X}_m$ to $\mathcal{M}_q(m,0;n+l,n)$ is bijective. Next, we define the relation $S_{(i,j)}$ on $\mathcal{X}_m$ to be the set of pairs $((\mathfrak{x},\xi),(\mathfrak{y},\eta))$ satisfying

$$\dim \mathfrak{x}\cap\mathfrak{y} = m-i \text{ and } \operatorname{rank}(\xi|_{\mathfrak{x}\cap\mathfrak{y}} - \eta|_{\mathfrak{x}\cap\mathfrak{y}}) = j.$$

Then, by the two equalities

$$\begin{aligned} \dim(\mathfrak{x}_\xi+\mathfrak{e})/\mathfrak{e} \cap (\mathfrak{y}_\eta+\mathfrak{e})/\mathfrak{e} &= \dim(\mathfrak{x}\cap\mathfrak{y}\oplus E)/(\{0\}\oplus E) \\ &= \dim \mathfrak{x}\cap\mathfrak{y} \end{aligned}$$

and

$$\begin{aligned} \dim \mathfrak{x}_\xi \cap \mathfrak{y}_\eta &= \dim\{(x,\xi(x)) \mid x \in \ker(\xi|_{\mathfrak{x}\cap\mathfrak{y}} - \eta|_{\mathfrak{x}\cap\mathfrak{y}})\} \\ &= \dim\ker(\xi|_{\mathfrak{x}\cap\mathfrak{y}} - \eta|_{\mathfrak{x}\cap\mathfrak{y}}) \\ &= \dim \mathfrak{x}\cap\mathfrak{y} - \operatorname{rank}(\xi|_{\mathfrak{x}\cap\mathfrak{y}} - \eta|_{\mathfrak{x}\cap\mathfrak{y}}), \end{aligned}$$

it follows that $((\mathfrak{x},\xi),(\mathfrak{y},\eta)) \in S_{(i,j)}$ if and only if $(\mathfrak{x}_\xi,\mathfrak{y}_\eta) \in R_{(i,j)}$. Consequently, we obtain another representation of $\mathfrak{X}(\mathcal{M}_q(m,0;n+l,n))$, and we denote the association scheme $(\mathcal{X}_m,\{S_{(i,j)}\}_{(i,j)\in K})$ by $\mathfrak{X}(\mathcal{M}_q(m,0;n+l,n))$.

We calculate the character table of this association scheme using the following theorem proved by Tarnanen, Aaltonen and Goethals [12]:

**Theorem 2.1.** *Let $X$ be a non-empty finite set. Assume that $\{R_i\}_{0\le i\le d}$ is a partition of $X\times X$, each $R_i$ is a symmetric relation, that is ${}^tR_i = R_i$, and $R_0 = \{(x,x)\,|\,x\in X\}$. Let $\mathfrak{A} = \langle A_0, A_1,\dots,A_d\rangle_{\mathbb{C}}$ be the complex linear space generated by the adjacency matrices $A_i$ of $R_i$ $(i\in\{0,1,\dots,d\})$. If there exist matrices $C_0 = J, C_1,\dots,C_d$ of $\mathfrak{A}$ such that*

$$A_i = \sum_{j=0}^{d} \alpha_i(j) C_j \ \text{ and } \ C_k C_s = \sum_{j=0}^{k\wedge s} \beta_{k;s}(j) C_j, \quad k,s\in\{0,1,\dots,d\},$$

*where $\alpha_i(j)$and $\beta_{k;s}(j)$ are complex numbers, then $(X,\{R_i\}_{0\le i\le d})$ is a symmetric association scheme and its first eigenmatrix $P = (P_i(s))_{0\le s,i\le d}$ is given by*

$$P_i(s) = \sum_{k=s}^{d} \alpha_i(k)\beta_{k;s}(s), \quad i,s\in\{0,1,\dots,d\}.$$

For the rest of this paper, let $d = m\wedge(n-m)$. Let $A_{(i,j)}$ denote the adjacency matrix of the relation $S_{(i,j)}$. We define $L = \{(r,s)\in {\mathbb{Z}_{\ge 0}}^2 \,|\, s\le m\wedge l, r\le m,\ 0\le r-s\le d\}$. Since $(i,j)\mapsto(r,s) = (i+j,j)$ is a bijection from $K$ to $L$, we have $|L| = |K|$. For $(r,s)\in L$, a matrix $C_{(r,s)}$ is defined as follows, which serves as $C_i$ in Theorem 2.1: for $0\le i\le d$ and $0\le s\le m\wedge l$,

$$B_{(i,s)} = \sum_{j=0}^{(m-i)\wedge l} K_s(m-i,l;q;j) A_{(i,j)}$$

and for $(r,s)\in L$,

$$C_{(r,s)} = \sum_{i=0}^{d\wedge(m-s)} \begin{bmatrix} m-i-s \\ r-s \end{bmatrix} B_{(i,s)}.$$

Namely, for $((\mathfrak{x},\xi),(\mathfrak{y},\eta)) \in S_{(m-u,u-v)}$, the $((\mathfrak{x},\xi),(\mathfrak{y},\eta))$-entry of $C_{(r,s)}$ is as follows:

$$C_{(r,s)}((\mathfrak{x},\xi),(\mathfrak{y},\eta)) = \begin{bmatrix} u-s \\ r-s \end{bmatrix} K_s(u,l;q;u-v). \tag{2.3}$$

By using (2.2) for the bilinear forms scheme $H_q(m-i,l)$, for $(i,j)\in K$, we obtain

$$\begin{aligned} A_{(i,j)} &= q^{-(m-i)l} \sum_{h=0}^{(m-i)\wedge l} K_j(m-i,l;q;h) B_{(i,h)} \\ &= q^{-(m-i)l} \sum_{h=0}^{m\wedge l} K_j(m-i,l;q;h) B_{(i,h)}, \end{aligned}$$

since $K_j(m-i,l;q;h) = 0$ if $h > m-i$. We claim that the square matrix

$$M^s = \left( \begin{bmatrix} m-i-s \\ r-s \end{bmatrix} \right)_{\substack{0\le i\le d\wedge(m-s),\\ s\le r\le (d+s)\wedge m}}$$

is nonsingular. Indeed, since $\begin{bmatrix} m-i-s \\ r-s \end{bmatrix}$ is a polynomial of degree $r-s$ in $q^{-i}$, $M^s$ can be converted into a Vandermonde matrix $((q^{-i})^{r'})_{0\le i,r'\le d\wedge(m-s)}$ by a sequence of elementary transformations. Thus the inverse matrix $N^s$ of $M^s$ satisfies

$$\sum_{k=s}^{(d+s)\wedge m} \begin{bmatrix} m-i-s \\ k-s \end{bmatrix} N^s(k,j) = \delta_{i,j} \tag{2.4}$$

for $0 \le i, j \le d \wedge (m-s)$. By (2.4), for $0 \le i \le d$ and $0 \le h \le m \wedge l$, we obtain

$$B_{(i,h)} = \sum_{k=h}^{(d+h)\wedge m} N^h(k,i)C_{(k,h)}.$$

Therefore we have $A_{(i,j)} = \sum_{(k,h)\in L} \alpha_{(i,j)}(k,h)C_{(k,h)}$, where

$$\alpha_{(i,j)}(k,h) = q^{-(m-i)l} K_j(m-i,l;q;h)N^h(k,i). \tag{2.5}$$

The following lemma gives the expansion of $C_{(r,s)}C_{(k,h)}$ in terms of $C_{(i,j)}$, which serves as the expansion of $C_kC_s$ in terms of $C_j$ in Theorem 2.1.

**Lemma 2.2.** *The matrices* $\{C_{(r,s)}\}_{(r,s)\in L}$ *satisfy*

$$C_{(r,s)}C_{(k,h)} = \sum_{i=s}^{r\wedge k} \beta_{(r,s;k,h)}(i,s)C_{(i,s)},$$

*where*

$$\beta_{(r,s;k,h)}(i,s) = \delta_{s,h} q^{ml+(i-s)(m-r-k+i)} \begin{bmatrix} m-i \\ m-r \end{bmatrix} \begin{bmatrix} m-i \\ m-k \end{bmatrix} \begin{bmatrix} n-r-k+s \\ m-r-k+i \end{bmatrix}.$$

We will prove Lemma 2.2 in Section 3. Using Lemma 2.2, we obtain the main theorem of this paper.

**Theorem 2.3.** *Let* $P = (P_{(i,j)}(r,s))_{(r,s)\in L,(i,j)\in K}$ *and* $Q = (Q_{(r,s)}(i,j))_{(i,j)\in K,(r,s)\in L}$ *be the first and second eigenmatrices of the association scheme based on attenuated space* $\mathfrak{X}(\mathcal{M}_q(m,0;n+l,n))$*, respectively. Then the following hold.*

$$P_{(i,j)}(r,s) = q^{il} K_j(m-i,l;q;s)E_i(n-s,m-s;q;r-s) \tag{2.6}$$

*and*

$$Q_{(r,s)}(i,j) = \frac{\begin{bmatrix} n \\ m \end{bmatrix}}{\begin{bmatrix} n-s \\ m-s \end{bmatrix}} K_s(m-i,l;q;j)Q_{r-s}(n-s,m-s;q;i). \tag{2.7}$$

*Proof.* Let $\succeq$ be the lexicographical order of $L$ so that the relation $(k,h) \succeq (r,s)$ means that $h > s$ or that $k \ge r$ if $h = s$. By Theorem 2.1, the eigenvalue $P_{(i,j)}(r,s)$ of $\mathfrak{X}(\mathcal{M}_q(m,0;n+l,n)) = (\mathcal{X}_m, \{S_{(i,j)}\}_{(i,j)\in K})$ is calculated as follows. For $(i,j) \in$

$K$ and $(r,s) \in L$, we have

$$\begin{aligned} P_{(i,j)}(r,s) &= \sum_{(k,h)\succeq(r,s)} \alpha_{(i,j)}(k,h)\beta_{(k,h;r,s)}(r,s) \\ &= \sum_{k=r}^{m\wedge(d+s)} q^{-(m-i)l} K_j(m-i,l;q;s) N^s(k,i) \\ &\quad \times q^{ml} q^{(r-s)(m-k)} \begin{bmatrix} m-r \\ m-k \end{bmatrix} \begin{bmatrix} n-k-r+s \\ m-k \end{bmatrix} \\ &= q^{il} K_j(m-i,l;q;s) \sum_{k=s}^{m\wedge(d+s)} N^s(k,i) \\ &\quad \times q^{(m-k)(r-s)} \begin{bmatrix} m-r \\ m-k \end{bmatrix} \begin{bmatrix} n-k-r+s \\ m-k \end{bmatrix}, \end{aligned}$$

since $\left[\begin{smallmatrix} m-r \\ m-k \end{smallmatrix}\right] = 0$ if $k < r$. Moreover using Lemma A.4 and (2.4), we obtain

$$\begin{aligned} P_{(i,j)}(r,s) &= q^{il} K_j(m-i,l;q;s) \sum_{k=s}^{m\wedge(d+s)} N^s(k,i) \\ &\quad \times \sum_{t=0}^{m-s} \begin{bmatrix} m-s-t \\ k-s \end{bmatrix} E_t(n-s,m-s;q;r-s) \\ &= q^{il} K_j(m-i,l;q;s) E_i(n-s,m-s;q;r-s). \end{aligned}$$

To verify (2.7), it is sufficient to show that

$$\sum_{(i,j)\in K} \frac{\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]}{\left[\begin{smallmatrix} n-s \\ m-s \end{smallmatrix}\right]} K_s(m-i,l;q;j) Q_{r-s}(n-s,m-s;q;i) P_{(i,j)}(k,h) = q^{ml} \begin{bmatrix} n \\ m \end{bmatrix} \delta_{r,k}\delta_{s,h}$$

for all $(r,s)$ and $(k,h)$ in $L$. Substituting (2.6) for $P_{(i,j)}(k,h)$ and using the orthogonality of eigenmatrices of Grassmann schemes $J_q(n-s,m-s)$ and bilinear forms schemes $H_q(m-i,l)$, this orthogonality relation is verified. $\square$

## 3. Proof of Lemma 2.2

Let $\mathcal{Y}$ be the set of all pairs of a subspace $\mathfrak{a}$ in $V$ and a linear map $f$ from $E$ to $\mathfrak{a}$, and $\mathcal{Y}_r = \{(\mathfrak{a}, f) \in \mathcal{Y} \mid \dim \mathfrak{a} = r\}$. For a subspace $\mathfrak{x}$ of $V$, let $\mathcal{Y}_r^{\mathfrak{x}} = \{(\mathfrak{a}, f) \in \mathcal{Y}_r \mid \mathfrak{a} \subset \mathfrak{x}\}$ and $\mathcal{Y}_{r,s}^{\mathfrak{x}} = \{(\mathfrak{a}, f) \in \mathcal{Y}_r^{\mathfrak{x}} \mid \operatorname{rank} f = s\}$. We shall define a complex-valued function $\langle \cdot, \cdot \rangle$ over $\mathcal{Y} \times \mathcal{X}$ needed in the proof of Lemma 2.2. Consider a pair $((\mathfrak{a}, f), (\mathfrak{x}, \xi)) \in \mathcal{Y} \times \mathcal{X}$ satisfying $\mathfrak{a} \subset \mathfrak{x}$. When $\dim \mathfrak{x} = r$, the linear map $f \circ \xi : \mathfrak{x} \to \mathfrak{a} \subset \mathfrak{x}$ is regarded as a square matrix $T_{f\circ\xi}$ of degree $r$ for some basis of $\mathfrak{x}$. Note that the trace $\operatorname{Tr}(T_{f\circ\xi})$ of $T_{f\circ\xi}$ is independent of the chosen basis of $\mathfrak{x}$, so that $\operatorname{Tr}(T_{f\circ\xi})$ only depends on the pair $((\mathfrak{a}, f), (\mathfrak{x}, \xi))$.

**Definition 3.1.** Let $p$ be the prime divisor of $q$ and let $\epsilon$ be a primitive $p$-th root of unity. We take $\chi : \mathbb{F}_q \to \mathbb{Z}[\epsilon]$ to be a non-principal character of the elementary abelian $p$-group $(\mathbb{F}_q, +)$. The function $\mathcal{Y} \times \mathcal{X} \ni ((\mathfrak{a}, f), (\mathfrak{x}, \xi)) \mapsto \langle (\mathfrak{a}, f), (\mathfrak{x}, \xi) \rangle \in \mathbb{C}$ is defined as

$$\langle (\mathfrak{a}, f), (\mathfrak{x}, \xi) \rangle = \begin{cases} \chi(\operatorname{Tr}(T_{f\circ\xi})) & \text{if } \mathfrak{a} \subset \mathfrak{x}, \\ 0 & \text{otherwise.} \end{cases}$$

Fix an element $\mathfrak{x}$ in $\begin{bmatrix} V \\ r \end{bmatrix}$. For $f \in L(E, \mathfrak{x})$, we define the function $\Lambda_f : L(\mathfrak{x}, E) \to \mathbb{C}$ by $\Lambda_f(\xi) = \langle (\mathfrak{x}, f), (\mathfrak{x}, \xi) \rangle$. Then $\{\Lambda_f \mid f \in L(E, \mathfrak{x})\}$ is the character group of $L(\mathfrak{x}, E)$ and we denote it by $L(\mathfrak{x}, E)^*$. There is the orthogonality relation of $L(\mathfrak{x}, E)^*$:

$$\sum_{\xi \in L(\mathfrak{x}, E)} \overline{\langle (\mathfrak{x}, f), (\mathfrak{x}, \xi) \rangle} \langle (\mathfrak{x}, g), (\mathfrak{x}, \xi) \rangle = \begin{cases} q^{rl} & \text{if } f = g, \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

and there is the following equation (cf. [8]):

$$\sum_{\substack{f \in L(E, \mathfrak{x}) \\ \operatorname{rank} f = s}} \langle (\mathfrak{x}, f), (\mathfrak{x}, \xi) \rangle \overline{\langle (\mathfrak{x}, f), (\mathfrak{x}, \eta) \rangle} = K_s(r, l; q; \operatorname{rank}(\xi - \eta)). \tag{3.2}$$

For $(\mathfrak{a}, f)$ and $(\mathfrak{b}, g)$ in $\mathcal{Y}$, if $f(x) = g(x)$ for any $x \in E$, then we call the pair $(\mathfrak{a}, f)$ and $(\mathfrak{b}, g)$ *almost equal* and denote this by $(\mathfrak{a}, f) \approx (\mathfrak{b}, g)$. When $(\mathfrak{a}, f) \approx (\mathfrak{b}, g)$, it can be easily seen that $\operatorname{Im} f = \operatorname{Im} g \subset \mathfrak{a} \cap \mathfrak{b}$ and $\langle (\mathfrak{a}, f), (\mathfrak{x}, \xi) \rangle = \langle (\mathfrak{b}, g), (\mathfrak{x}, \xi) \rangle$ for any $(\mathfrak{x}, \xi) \in \mathcal{X}$ with $\mathfrak{a} + \mathfrak{b} \subset \mathfrak{x}$.

**Lemma 3.2.** *For $(\mathfrak{x}, \xi)$ and $(\mathfrak{y}, \eta)$ in $\mathcal{X}_m$,*

$$\sum_{(\mathfrak{a}, f) \in \mathcal{Y}_{r,s}^V} \langle (\mathfrak{a}, f), (\mathfrak{x}, \xi) \rangle \overline{\langle (\mathfrak{a}, f), (\mathfrak{y}, \eta) \rangle} = C_{(r,s)}((\mathfrak{x}, \xi), (\mathfrak{y}, \eta)). \tag{3.3}$$

*Proof.* By the definition of $\langle \cdot, \cdot \rangle$, the range of the summation of $(\mathfrak{a}, f)$ in the left hand side of (3.3) is restricted within $\mathcal{Y}_{r,s}^{\mathfrak{x} \cap \mathfrak{y}}$. Since $f \circ \xi$ has a matrix representation of the following form

$$T_{f \circ \xi} = \begin{pmatrix} T & * \\ 0 & 0 \end{pmatrix},$$

where $T$ is a matrix representation of $f \circ \xi|_{\mathfrak{a}} \in L(\mathfrak{a}, \mathfrak{a})$, we obtain

$$\langle (\mathfrak{a}, f), (\mathfrak{x}, \xi) \rangle = \langle (\mathfrak{a}, f), (\mathfrak{a}, \xi|_{\mathfrak{a}}) \rangle.$$

Similarly, we obtain

$$\langle (\mathfrak{a}, f), (\mathfrak{y}, \eta) \rangle = \langle (\mathfrak{a}, f), (\mathfrak{a}, \eta|_{\mathfrak{a}}) \rangle.$$

Therefore by (3.2), we obtain

$$\begin{aligned} \sum_{(\mathfrak{a}, f) \in \mathcal{Y}_{r,s}^V} \langle (\mathfrak{a}, f), (\mathfrak{x}, \xi) \rangle \overline{\langle (\mathfrak{a}, f), (\mathfrak{y}, \eta) \rangle} &= \sum_{(\mathfrak{a}, f) \in \mathcal{Y}_{r,s}^{\mathfrak{x} \cap \mathfrak{y}}} \langle (\mathfrak{a}, f), (\mathfrak{a}, \xi|_{\mathfrak{a}}) \rangle \overline{\langle (\mathfrak{a}, f), (\mathfrak{a}, \eta|_{\mathfrak{a}}) \rangle} \\ &= \sum_{\mathfrak{a} \in \left[ {\mathfrak{x} \cap \mathfrak{y} \atop r} \right]} K_s(r, l; q; \operatorname{rank}(\xi|_{\mathfrak{a}} - \eta|_{\mathfrak{a}})). \end{aligned}$$

Let us remark that $\operatorname{rank}(\xi|_{\mathfrak{a}} - \eta|_{\mathfrak{a}}) = \dim \mathfrak{a} - \dim \ker(\xi|_{\mathfrak{a}} - \eta|_{\mathfrak{a}})$ and $\ker(\xi|_{\mathfrak{a}} - \eta|_{\mathfrak{a}}) = \ker(\xi|_{\mathfrak{x} \cap \mathfrak{y}} - \eta|_{\mathfrak{x} \cap \mathfrak{y}}) \cap \mathfrak{a}$. Thus when $\dim \mathfrak{x} \cap \mathfrak{y} = u$ and $\dim \ker(\xi|_{\mathfrak{x} \cap \mathfrak{y}} - \eta|_{\mathfrak{x} \cap \mathfrak{y}}) = v$ (i.e., $((\mathfrak{x}, \xi), (\mathfrak{y}, \eta)) \in S_{(m-u, u-v)}$), by Corollary A.2, the number of $\mathfrak{a} \in \left[ {\mathfrak{x} \cap \mathfrak{y} \atop r} \right]$ with $\dim \ker(\xi|_{\mathfrak{x} \cap \mathfrak{y}} - \eta|_{\mathfrak{x} \cap \mathfrak{y}}) \cap \mathfrak{a} = r - j$ is equal to

$$q^{j(v-r+j)} \begin{bmatrix} u - v \\ j \end{bmatrix} \begin{bmatrix} v \\ r - j \end{bmatrix}.$$

Hence it follows that

$$\begin{aligned}\sum_{\mathfrak{a}\in\left[{\mathfrak{x}\cap\mathfrak{y} \atop r}\right]} K_s(r,l;q;\operatorname{rank}(\xi|_{\mathfrak{a}}-\eta|_{\mathfrak{a}})) &= \sum_{j=0\vee(v-r)}^{r\wedge(u-v)} q^{j(v-r+j)}\begin{bmatrix}u-v\\ j\end{bmatrix}\begin{bmatrix}v\\ r-j\end{bmatrix}K_s(r,l;q;j)\\ &= \begin{bmatrix}u-s\\ r-s\end{bmatrix}K_s(u,l;q;u-v)\\ &= C_{(r,s)}((\mathfrak{x},\xi),(\mathfrak{y},\eta)),\end{aligned}$$

where for integers $a$ and $b$, the value $a\vee b$ denotes $\max\{a,b\}$ for short. In the second line, we used Lemma A.5. □

*Proof of Lemma 2.2.* Since $\mathfrak{X}(\mathcal{M}_q(m,0;n+l,n))$ is symmetric, without loss of generality, we may suppose $r\le k$. By Lemma 3.2, for $((\mathfrak{x},\xi),(\mathfrak{y},\eta))\in S_{(m-u,u-v)}$, we obtain

$$\begin{aligned}C_{(r,s)}C_{(k,h)}((\mathfrak{x},\xi),(\mathfrak{y},\eta)) &= \sum_{(\mathfrak{z},\zeta)\in\mathcal{X}_m} C_{(r,s)}((\mathfrak{x},\xi),(\mathfrak{z},\zeta))\cdot C_{(k,h)}((\mathfrak{z},\zeta),(\mathfrak{y},\eta))\\ &= \sum_{(\mathfrak{a},f)\in\mathcal{Y}^V_{r,s}}\ \sum_{(\mathfrak{b},g)\in\mathcal{Y}^V_{k,h}} \langle(\mathfrak{a},f),(\mathfrak{x},\xi)\rangle\overline{\langle(\mathfrak{b},g),(\mathfrak{y},\eta)\rangle}\\ &\quad\times\sum_{(\mathfrak{z},\zeta)\in\mathcal{X}_m}\overline{\langle(\mathfrak{a},f),(\mathfrak{z},\zeta)\rangle}\langle(\mathfrak{b},g),(\mathfrak{z},\zeta)\rangle.\end{aligned}$$

By the definition of $\langle\cdot,\cdot\rangle$, we may restrict the range of the summation to be those $(\mathfrak{z},\zeta)\in\mathcal{X}_m$ satisfying $\mathfrak{z}\supset\mathfrak{a}+\mathfrak{b}$. Thus we obtain

$$\begin{aligned}\sum_{(\mathfrak{z},\zeta)\in\mathcal{X}_m}\overline{\langle(\mathfrak{a},f),(\mathfrak{z},\zeta)\rangle}\langle(\mathfrak{b},g),(\mathfrak{z},\zeta)\rangle &= \sum_{\substack{\mathfrak{z}\in\left[{V\atop m}\right]\\ \mathfrak{z}\supset\mathfrak{a}+\mathfrak{b}}}\ \sum_{\zeta\in L(\mathfrak{z},E)}\overline{\langle(\mathfrak{a},f),(\mathfrak{z},\zeta)\rangle}\langle(\mathfrak{b},g),(\mathfrak{z},\zeta)\rangle\\ &= \sum_{\substack{\mathfrak{z}\in\left[{V\atop m}\right]\\ \mathfrak{z}\supset\mathfrak{a}+\mathfrak{b}}}\ \sum_{\zeta\in L(\mathfrak{z},E)}\overline{\langle(\mathfrak{z},f),(\mathfrak{z},\zeta)\rangle}\langle(\mathfrak{z},g),(\mathfrak{z},\zeta)\rangle\end{aligned}$$

and, by (3.1), the value $\sum_{\zeta\in L(\mathfrak{z},E)}\overline{\langle(\mathfrak{z},f),(\mathfrak{z},\zeta)\rangle}\langle(\mathfrak{z},g),(\mathfrak{z},\zeta)\rangle$ is equal to $q^{ml}$ if $(\mathfrak{a},f)\approx(\mathfrak{b},g)$ or vanishes otherwise. Hence, if $h\neq s$, then $C_{(r,s)}C_{(k,h)}=0$ and we will henceforth assume that $h=s$. The number of $\mathfrak{z}\in\left[{V\atop m}\right]$ satisfying $\mathfrak{z}\supset\mathfrak{a}+\mathfrak{b}$ is equal to $\left[{n-\dim(\mathfrak{a}+\mathfrak{b})\atop m-\dim(\mathfrak{a}+\mathfrak{b})}\right]$. Thus we obtain

$$\begin{aligned}&C_{(r,s)}C_{(k,s)}((\mathfrak{x},\xi),(\mathfrak{y},\eta))\\ &\quad= \sum_{\substack{(\mathfrak{a},f)\in\mathcal{Y}^{\mathfrak{x}}_{r,s}\\ (\mathfrak{b},g)\in\mathcal{Y}^{\mathfrak{y}}_{k,s}\\ (\mathfrak{a},f)\approx(\mathfrak{b},g)}} \langle(\mathfrak{a},f),(\mathfrak{x},\xi)\rangle\overline{\langle(\mathfrak{b},g),(\mathfrak{y},\eta)\rangle}q^{ml}\begin{bmatrix}n-\dim(\mathfrak{a}+\mathfrak{b})\\ m-\dim(\mathfrak{a}+\mathfrak{b})\end{bmatrix}.\end{aligned}\tag{3.4}$$

Let us impose the additional constraints: $\dim\mathfrak{a}\cap\mathfrak{b}=e$, $\dim\mathfrak{a}\cap\mathfrak{y}=r-t$ and $\dim\mathfrak{b}\cap\mathfrak{x}=k-j$. Since $(\mathfrak{a},f)\approx(\mathfrak{b},g)$, i.e., $\operatorname{Im}f=\operatorname{Im}g\subset\mathfrak{a}\cap\mathfrak{b}$, it follows that $e\ge s$ and that $C_{(r,s)}C_{(k,h)}((\mathfrak{x},\xi),(\mathfrak{y},\eta))$ is equal to

$$q^{ml}\sum_{e=s}^{r}\begin{bmatrix}n-r-k+e\\ m-r-k+e\end{bmatrix}\sum_{t=0}^{r-e}\sum_{j=0}^{k-e}\sum_{(\mathfrak{a},f)\in T_1}\sum_{\mathfrak{b}\in T_2}\langle(\mathfrak{a},f),(\mathfrak{x},\xi)\rangle\overline{\langle(\mathfrak{b},f),(\mathfrak{y},\eta)\rangle},$$

where $T_1 = \{(\mathfrak{a}, f) \in \mathcal{Y}^{\mathfrak{x}}_{r,s} \mid \dim \mathfrak{a}\cap\mathfrak{y} = r-t,\ \mathrm{Im}\, f \subset \mathfrak{a}\cap\mathfrak{y}\}$ and $T_2 = \{\mathfrak{b} \in \begin{bmatrix}\mathfrak{y}\\ k\end{bmatrix} \mid \dim \mathfrak{b}\cap \mathfrak{x} = k-j,\ \dim \mathfrak{a}\cap\mathfrak{b} = e,\ \mathfrak{b} \supset \mathrm{Im}\, f\}$. Since $\langle(\mathfrak{b}, f), (\mathfrak{y}, \eta)\rangle = \langle(\mathfrak{a}\cap\mathfrak{y}, f), (\mathfrak{y}, \eta)\rangle$ is independent of $\mathfrak{b}$, it follows that

$$\begin{aligned}(3.5)\qquad &\sum_{(\mathfrak{a},f)\in T_1}\sum_{\mathfrak{b}\in T_2}\langle(\mathfrak{a}, f), (\mathfrak{x}, \xi)\rangle\overline{\langle(\mathfrak{b}, f), (\mathfrak{y}, \eta)\rangle}\\ &= \sum_{(\mathfrak{a},f)\in T_1}|T_2|\langle(\mathfrak{a}, f), (\mathfrak{x}, \xi)\rangle\overline{\langle(\mathfrak{a}\cap\mathfrak{y}, f), (\mathfrak{y}, \eta)\rangle}.\end{aligned}$$

We shall count the number of elements in $T_2$. By Proposition A.1, the number of $\mathfrak{b}' \in \begin{bmatrix}\mathfrak{x}\cap\mathfrak{y}\\ k-j\end{bmatrix}$ satisfying $\mathfrak{b}' \supset \mathrm{Im}\, f$ and $\dim \mathfrak{a}\cap\mathfrak{b}' = e$ is

$$q^{(r-t-e)(k-j-e)}\begin{bmatrix}r-t-s\\ e-s\end{bmatrix}\begin{bmatrix}u-r+t\\ k-j-e\end{bmatrix}$$

and the number of $\mathfrak{b} \in \begin{bmatrix}\mathfrak{y}\\ k\end{bmatrix}$ satisfying $\mathfrak{b}\cap\mathfrak{x} = \mathfrak{b}'$ is

$$q^{j(u-k+j)}\begin{bmatrix}m-u\\ j\end{bmatrix},$$

that is,

$$(3.6)\qquad |T_2| = q^{(r-t-e)(k-j-e)+j(u-k+j)}\begin{bmatrix}r-t-s\\ e-s\end{bmatrix}\begin{bmatrix}u-r+t\\ k-j-e\end{bmatrix}\begin{bmatrix}m-u\\ j\end{bmatrix}.$$

This means that $|T_2|$ is independent of the choice of $(\mathfrak{a}, f)$. Note that, for $(\mathfrak{a}', f') \in \mathcal{Y}^{\mathfrak{x}\cap\mathfrak{y}}_{r-t,s}$, all $(\mathfrak{a}, f) \in T_1$ with $\mathfrak{a}\cap\mathfrak{y} = \mathfrak{a}'$ and $f(x) = f'(x)$ for any $x \in E$ satisfy $\langle(\mathfrak{a}, f), (\mathfrak{x}, \xi)\rangle\overline{\langle(\mathfrak{a}\cap\mathfrak{y}, f), (\mathfrak{y}, \eta)\rangle} = \langle(\mathfrak{a}', f'), (\mathfrak{x}, \xi)\rangle\overline{\langle(\mathfrak{a}', f'), (\mathfrak{y}, \eta)\rangle}$. On the other hand, by Proposition A.1, it follows that the number of $(\mathfrak{a}, f) \in T_1$ with $\mathfrak{a}\cap\mathfrak{y} = \mathfrak{a}'$ and $f(x) = f'(x)$ for any $x \in E$ is

$$q^{t(u-r+t)}\begin{bmatrix}m-u\\ t\end{bmatrix}.$$

Therefore we obtain

$$\begin{aligned}(3.7)\qquad &\sum_{(\mathfrak{a},f)\in T_1}\langle(\mathfrak{a}, f), (\mathfrak{x}, \xi)\rangle\overline{\langle(\mathfrak{a}\cap\mathfrak{y}, f), (\mathfrak{y}, \eta)\rangle}\\ &= q^{t(u-r+t)}\begin{bmatrix}m-u\\ t\end{bmatrix}\sum_{(\mathfrak{a}',f')\in\mathcal{Y}^{\mathfrak{x}\cap\mathfrak{y}}_{r-t,s}}\langle(\mathfrak{a}', f'), (\mathfrak{x}, \xi)\rangle\overline{\langle(\mathfrak{a}', f'), (\mathfrak{y}, \eta)\rangle}.\end{aligned}$$

By (3.5), (3.6) and (3.7) and Lemma 3.2, it follows that $C_{(r,s)}C_{(k,h)}((\mathfrak{x}, \xi), (\mathfrak{y}, \eta))$ is equal to

$$\begin{aligned}&q^{ml}\sum_{e=s}^{r}\begin{bmatrix}n-r-k+e\\ m-r-k+e\end{bmatrix}\sum_{t=0}^{r-e}\sum_{j=0}^{k-e}q^{(u-r+t)t+(u-k+j)j+(r-t-e)(k-j-e)}\\ &\times\begin{bmatrix}m-u\\ t\end{bmatrix}\begin{bmatrix}m-u\\ j\end{bmatrix}\begin{bmatrix}r-t-s\\ e-s\end{bmatrix}\begin{bmatrix}u-r+t\\ k-j-e\end{bmatrix}C_{(r-t,s)}((\mathfrak{x}, \xi), (\mathfrak{y}, \eta)).\end{aligned}$$

Substituting (2.3) for $C_{(r-t,s)}((\mathfrak{x}, \xi), (\mathfrak{y}, \eta))$ and using Proposition A.3 (4) (a) as $(k, x, r, t) = (m-s, u-s, r-s, r-t-s)$, i.e.,

$$\begin{bmatrix}u-s\\ r-t-s\end{bmatrix}\begin{bmatrix}m-u\\ t\end{bmatrix} = \sum_{i=s}^{r}(-1)^{i-r+t}q^{-t(u-r+t)+\binom{i-r+t}{2}}\begin{bmatrix}i-s\\ r-t-s\end{bmatrix}\begin{bmatrix}m-i\\ r-i\end{bmatrix}\begin{bmatrix}u-s\\ i-s\end{bmatrix},$$

and substituting (2.3) for $C_{(i,s)}((\mathfrak{x},\xi),(\mathfrak{y},\eta))$ again, $C_{(r,s)}C_{(k,h)}((\mathfrak{x},\xi),(\mathfrak{y},\eta))$ is equal to

$$q^{ml}\sum_{i=s}^{r}\sum_{e=s}^{r}\begin{bmatrix}n-r-k+e\\m-r-k+e\end{bmatrix}\sum_{t=0}^{r-e}(-1)^{i-r+t}\begin{bmatrix}r-t-s\\e-s\end{bmatrix}\begin{bmatrix}i-s\\r-t-s\end{bmatrix}\begin{bmatrix}m-i\\r-i\end{bmatrix}$$
$$\times q^{(r-t-e)(k-e)+\binom{i-r+t}{2}}\sum_{j=0}^{k-e}q^{j(u-r+t-k+j+e)}\begin{bmatrix}m-u\\j\end{bmatrix}\begin{bmatrix}u-r+t\\k-j-e\end{bmatrix}C_{(i,s)}((\mathfrak{x},\xi),(\mathfrak{y},\eta)).$$

Using Proposition A.3 (3) (b) for the summation of $j$, i.e.,

$$\sum_{j=0}^{k-e}q^{j(u-r+t-k+j+e)}\begin{bmatrix}m-u\\j\end{bmatrix}\begin{bmatrix}u-r+t\\k-j-e\end{bmatrix}=\begin{bmatrix}m-r+t\\k-e\end{bmatrix},$$

using Proposition A.3 (4) (b) as $(k,x,r,t)=(m-s,i-s,m-s-k+e,e-s)$, i.e.,

$$\sum_{t=0}^{r-e}(-1)^{i-r+t}q^{(r-t-e)(k-e)+\binom{i-r+t}{2}}\begin{bmatrix}r-t-s\\e-s\end{bmatrix}\begin{bmatrix}i-s\\r-t-s\end{bmatrix}\begin{bmatrix}m-r+t\\k-e\end{bmatrix}$$
$$=(-1)^{i-e}q^{\binom{i-e}{2}}\begin{bmatrix}i-s\\e-s\end{bmatrix}\begin{bmatrix}m-i\\m-k\end{bmatrix},$$

and using Proposition A.3 (4) (a) as $(k,x,r,t)=(n-r-k+i,i-s,m-r-k+i,0)$, i.e.,

$$\sum_{e=s}^{r}(-1)^{i-e}q^{\binom{i-e}{2}}\begin{bmatrix}n-r-k+e\\m-r-k+e\end{bmatrix}\begin{bmatrix}i-s\\e-s\end{bmatrix}=\begin{bmatrix}n-r-k+s\\m-r-k+i\end{bmatrix}q^{(m-r-k+i)(i-s)},$$

we have

$$\sum_{e=s}^{r}\begin{bmatrix}n-r-k+e\\m-r-k+e\end{bmatrix}\sum_{t=0}^{r-e}(-1)^{i-r+t}\begin{bmatrix}r-t-s\\e-s\end{bmatrix}\begin{bmatrix}i-s\\r-t-s\end{bmatrix}$$
$$\times q^{(r-t-e)(k-e)+\binom{i-r+t}{2}}\sum_{j=0}^{k-e}q^{j(u-r+t-k+j+e)}\begin{bmatrix}m-u\\j\end{bmatrix}\begin{bmatrix}u-r+t\\k-j-e\end{bmatrix}$$
$$=\begin{bmatrix}m-i\\m-k\end{bmatrix}\begin{bmatrix}n-r-k+s\\m-r-k+i\end{bmatrix}q^{(m-r-k+i)(i-s)}.$$

Therefore the desired result follows. □

## 4. Relations between association schemes based on attenuated spaces and $m$-flat association schemes

Assume that integers $n$ and $m$ satisfy $0<m<n$. First in this section, we describe the definition of $m$-flat association schemes. The cosets of $\mathbb{F}_q^n$ relative to any $m$-dimensional vector subspace are called *$m$-flats*. Let $X_n(m)$ be the set of all $m$-flats of $\mathbb{F}_q^n$. Then the cardinality of $X_n(m)$ is $q^{n-m}\left[\begin{smallmatrix}n\\m\end{smallmatrix}\right]$. We define the relation $T_{(i,j)}$ on $X_n(m)$ to be the set of pairs $(\mathfrak{p}+x,\mathfrak{q}+y)$ satisfying $\dim\mathfrak{p}\cap\mathfrak{q}=m-i$, for $0\le i\le d$, and $x-y\in\mathfrak{p}+\mathfrak{q}$ if $j=0$, $x-y\notin\mathfrak{p}+\mathfrak{q}$ if $j=1$. Then the pair $\tilde{J}_q(n,m)=(X_n(m),\{T_{(i,j)}\}_{0\le i\le d,0\le j\le(n-m-i)\wedge 1})$ is a symmetric association scheme and called an *$m$-flat association scheme*. Zhu and Li computed all intersection numbers of $m$-flat association schemes [14] and the author gave the character table of $m$-flat association schemes [10].

There are relations between $m$-flat association schemes and association schemes based on attenuated spaces as follows:

**Theorem 4.1.** *The $m$-flat association scheme $\tilde{J}_q(n,m)$ is isomorphic to the association scheme based on attenuated space $\mathfrak{X}(\mathcal{M}_q(n-m,0;n+1,n))$.*

We note that if $l=1$, the attenuated space is just an affine space, and Theorem 4.1 implies that Theorem 2.3 contains the main result of the paper [10] as a special case.

*Proof of Theorem 4.1.* Let $\mathscr{L}(x_1,x_2,\ldots,x_r)$ denote the subspace of $\mathbb{F}_q^n$ spanned by vectors $x_1,x_2,\ldots,x_r\in\mathbb{F}_q^n$. Let $e_i$ be the $i$-th standard base of $\mathbb{F}_q^n$ with 1 in the $i$-th component and 0 elsewhere, and $(\cdot,\cdot)$ be the standard non-degenerate symmetric bilinear form on $\mathbb{F}_q^n$ (i.e., $(\sum_{i=1}^n c_ie_i,\sum_{i=1}^n c_i'e_i)=\sum_{i=1}^n c_ic_i'$). For a subspace $\mathfrak{p}$ of $\mathbb{F}_q^n$, we define $\mathfrak{p}^\perp=\{\beta\in\mathbb{F}_q^n\,|\,(\alpha,\beta)=0\ (\forall\alpha\in\mathfrak{x})\}$. We regard $\mathbb{F}_q^n\oplus\mathbb{F}_q$ as $\mathbb{F}_q^{n+1}$ and also regard $\{0\}\oplus\mathbb{F}_q$ as $\mathfrak{e}$. For $\mathfrak{p}+x$ in $X_n(m)$, we set $\mathfrak{p}_x=\{(v,(x,v))\,|\,v\in\mathfrak{p}^\perp\}$ in $\mathbb{F}_q^{n+1}$. Since any $x$ and $x'$ in $\mathfrak{p}+x$ satisfy $(x,v)=(x',v)$, the map $\Phi:\mathfrak{p}+x\mapsto\mathfrak{p}_x$ from $X_n(m)$ to $\mathbb{F}_q^{n+1}$ is well-defined. Then we immediately check that $\dim\mathfrak{p}_x=\dim\mathfrak{p}^\perp=n-m$ and $\mathfrak{p}_x\cap\mathfrak{e}=\{(0,0)\}$, i.e., $\mathfrak{p}_x\in\mathcal{M}_q(n-m,0;n+1,n)$, and $\Phi$ is an injective map from $X_n(m)$ to $\mathcal{M}_q(n-m,0;n+1,n)$. On the other hand, we have $|X_n(m)|=|\mathcal{M}_q(n-m,0;n+1,n)|=q^{n-m}\left[{n \atop m}\right]$. This means that $\Phi$ is bijective.

Finally, we check that for each $\mathfrak{p}+x,\mathfrak{q}+y\in X_n(m)$, $(\mathfrak{p}+x,\mathfrak{q}+y)\in T_{(i,j)}$ if and only if $(\mathfrak{p}_x,\mathfrak{q}_y)\in R_{(i,j)}$. By

$$\begin{aligned}\dim(\mathfrak{p}_x+\mathfrak{e})/\mathfrak{e}\cap(\mathfrak{q}_y+\mathfrak{e})/\mathfrak{e} &= \dim\mathfrak{p}^\perp\cap\mathfrak{q}^\perp\\ &= n-\dim(\mathfrak{p}+\mathfrak{q}),\end{aligned}$$

it follows that $\dim\mathfrak{p}\cap\mathfrak{q}=m-i$ if and only if $\dim(\mathfrak{p}_x+\mathfrak{e})/\mathfrak{e}\cap(\mathfrak{q}_y+\mathfrak{e})/\mathfrak{e}=n-m-i$. Moreover, we have

$$\begin{aligned}\dim\mathfrak{p}_x\cap\mathfrak{q}_y &= \dim\{(v,(x,v))\,|\,v\in\mathfrak{p}^\perp\cap\mathfrak{q}^\perp\text{ and }(x,v)=(y,v)\}\\ &= \begin{cases}\dim\mathfrak{p}^\perp\cap\mathfrak{q}^\perp & \text{if }\mathfrak{p}^\perp\cap\mathfrak{q}^\perp\subset\mathscr{L}(x-y)^\perp,\\ \dim\mathfrak{p}^\perp\cap\mathfrak{q}^\perp-1 & \text{otherwise}\end{cases}\\ &= \begin{cases}n-m-i & \text{if }x-y\in\mathfrak{p}+\mathfrak{q},\\ n-m-i-1 & \text{otherwise.}\end{cases}\end{aligned}$$

Therefore the desired result follows. ☐

## Appendix A. Some formulas

**Proposition A.1** (cf. [9])**.** *Let $\mathfrak{p},\mathfrak{q}$ be subspaces of $\mathbb{F}_q^n$ with $\dim\mathfrak{p}=a$, $\dim\mathfrak{q}=b$, $\dim\mathfrak{p}\cap\mathfrak{q}=x$. For $c,y\in\mathbb{Z}$ with $x\le y\le a$, $b\le c\le n-a+y$, the number of subspaces $\mathfrak{w}$ of $\mathbb{F}_q^n$ with $\mathfrak{w}\supset\mathfrak{q}$, $\dim\mathfrak{w}=c$ and $\dim\mathfrak{w}\cap\mathfrak{p}=y$ is*

$$q^{(a-y)(c-b-y+x)}\begin{bmatrix}a-x\\ y-x\end{bmatrix}\begin{bmatrix}n-b-a+x\\ c-b-y+x\end{bmatrix}.$$

**Corollary A.2** (cf. [4, Lemma 9.3.2 (iii)])**.** *If $\mathfrak{p}$ is an $a$-dimensional subspace of $\mathbb{F}_q^n$, then there are precisely $q^{(a-y)(c-y)}\left[{a \atop y}\right]\left[{n-a \atop c-y}\right]$ $c$-dimensional subspaces $\mathfrak{w}$ of $\mathbb{F}_q^n$ with $\dim\mathfrak{p}\cap\mathfrak{w}=y$.*

**Proposition A.3.** (1) (a) $\begin{bmatrix}k\\ r\end{bmatrix}\begin{bmatrix}r\\ t\end{bmatrix}=\begin{bmatrix}k\\ t\end{bmatrix}\begin{bmatrix}k-t\\ r-t\end{bmatrix}$

(b) $\begin{bmatrix} k \\ r \end{bmatrix} \begin{bmatrix} r \\ t \end{bmatrix} = \begin{bmatrix} k \\ r-t \end{bmatrix} \begin{bmatrix} k-r+t \\ t \end{bmatrix}$

(2) $\displaystyle\sum_{i=r}^{k} (-1)^{k-i} q^{\binom{k-i}{2}} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} i \\ r \end{bmatrix} = \delta_{k,r}$

(3) (a) $\begin{bmatrix} x+y \\ k \end{bmatrix} = \displaystyle\sum_{i=0}^{k} q^{(x-i)(k-i)} \begin{bmatrix} x \\ i \end{bmatrix} \begin{bmatrix} y \\ k-i \end{bmatrix}$

(b) $\begin{bmatrix} x+y \\ k \end{bmatrix} = \displaystyle\sum_{i=0}^{k} q^{i(y-k+i)} \begin{bmatrix} x \\ i \end{bmatrix} \begin{bmatrix} y \\ k-i \end{bmatrix}$

(4) (a) $\begin{bmatrix} x \\ t \end{bmatrix} \begin{bmatrix} k-x \\ r-t \end{bmatrix} = \displaystyle\sum_{i=t}^{r} (-1)^{i-t} q^{-(r-t)(x-t)+\binom{i-t}{2}} \begin{bmatrix} i \\ t \end{bmatrix} \begin{bmatrix} k-i \\ r-i \end{bmatrix} \begin{bmatrix} x \\ i \end{bmatrix}$

(b) $\begin{bmatrix} x \\ t \end{bmatrix} \begin{bmatrix} k-x \\ r-t \end{bmatrix} = \displaystyle\sum_{i=t}^{r} (-1)^{i-t} q^{(i-t)(k-x-r+t)+\binom{i-t+1}{2}} \begin{bmatrix} i \\ t \end{bmatrix} \begin{bmatrix} k-i \\ r-i \end{bmatrix} \begin{bmatrix} x \\ i \end{bmatrix}$

*Proof.* (1) Immediate from the definition of the Gaussian coefficient.

(2) and (3) are well known (cf. [5] and [1, Theorem 3.4]).

(4) First, to prove Proposition A.3 (4) (a) and (b), we use the following claim. Remark that the equality between the first term and the second term of the claim is also proved by Lv and Wang [11].

**Claim.**

$$\begin{bmatrix} n-p \\ m \end{bmatrix} = \sum_{k=0}^{m} (-1)^k q^{-mp+\binom{k}{2}} \begin{bmatrix} p \\ k \end{bmatrix} \begin{bmatrix} n-k \\ m-k \end{bmatrix} = \sum_{k=0}^{m} (-1)^k q^{(n-m-p)k+\binom{k+1}{2}} \begin{bmatrix} p \\ k \end{bmatrix} \begin{bmatrix} n-k \\ m-k \end{bmatrix}$$

*Proof.* There is the following equation (cf. [11]):

$$\begin{bmatrix} n-p \\ m \end{bmatrix} = (-1)^m q^{m(n-p)-\binom{m}{2}} \begin{bmatrix} p-n+m-1 \\ m \end{bmatrix}. \tag{A.1}$$

By Proposition A.3 (3) (a), we have

$$\begin{bmatrix} p-n+m-1 \\ m \end{bmatrix} = \sum_{k=0}^{m} q^{(-n+m-1-k)(m-k)} \begin{bmatrix} -n+m-1 \\ k \end{bmatrix} \begin{bmatrix} p \\ m-k \end{bmatrix}.$$

Again, we apply (A.1) for $\left[\begin{smallmatrix} -n+m-1 \\ k \end{smallmatrix}\right]$ as $\left[\begin{smallmatrix} -n+m-1 \\ k \end{smallmatrix}\right] = (-1)^k q^{k(-n+m-1)-\binom{k}{2}} \left[\begin{smallmatrix} n-m+k \\ k \end{smallmatrix}\right]$. Consequently, it follows that

$$\begin{aligned} \begin{bmatrix} n-p \\ m \end{bmatrix} &= (-1)^m q^{m(n-p)-\binom{m}{2}} \sum_{k=0}^{m} q^{(-n+m-1-k)(m-k)} \begin{bmatrix} p \\ m-k \end{bmatrix} \\ &\quad \times (-1)^k q^{k(-n+m-1)-\binom{k}{2}} \begin{bmatrix} n-m+k \\ k \end{bmatrix} \\ &= \sum_{k=0}^{m} (-1)^{m-k} q^{-mp+\binom{m-k}{2}} \begin{bmatrix} n-m+k \\ k \end{bmatrix} \begin{bmatrix} p \\ m-k \end{bmatrix} \\ &= \sum_{k=0}^{m} (-1)^k q^{-mp+\binom{k}{2}} \begin{bmatrix} n-k \\ m-k \end{bmatrix} \begin{bmatrix} p \\ k \end{bmatrix}. \end{aligned}$$

Similarly, using Proposition A.3 (3) (b) instead of Proposition A.3 (3) (a), we obtain $\left[\begin{smallmatrix} n-p \\ m \end{smallmatrix}\right] = \sum_{k=0}^{m} (-1)^k q^{(n-m-p)k+\binom{k+1}{2}} \left[\begin{smallmatrix} p \\ k \end{smallmatrix}\right] \left[\begin{smallmatrix} n-k \\ m-k \end{smallmatrix}\right]$. □

Let us return to the proof of Proposition A.3 (4) (a). Using the above claim, we have

$$\begin{aligned}
\begin{bmatrix} k-x \\ r-t \end{bmatrix} &= \begin{bmatrix} (k-t)-(x-t) \\ r-t \end{bmatrix} \\
&= \sum_{i'=0}^{r-t} (-1)^{i'} q^{-(r-t)(x-t)+\binom{i'}{2}} \begin{bmatrix} k-t-i' \\ r-t-i' \end{bmatrix} \begin{bmatrix} x-t \\ i' \end{bmatrix} \\
&= \sum_{i=t}^{r} (-1)^{i-t} q^{-(r-t)(x-t)+\binom{i-t}{2}} \begin{bmatrix} k-i \\ r-i \end{bmatrix} \begin{bmatrix} x-t \\ i-t \end{bmatrix}.
\end{aligned}$$

Therefore it follows that

$$\begin{aligned}
\begin{bmatrix} x \\ t \end{bmatrix} \begin{bmatrix} k-x \\ r-t \end{bmatrix} &= \sum_{i=t}^{r} (-1)^{i-t} q^{-(r-t)(x-t)+\binom{i-t}{2}} \begin{bmatrix} k-i \\ r-i \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \begin{bmatrix} x-t \\ i-t \end{bmatrix} \\
&= \sum_{i=t}^{r} (-1)^{i-t} q^{-(r-t)(x-t)+\binom{i-t}{2}} \begin{bmatrix} k-i \\ r-i \end{bmatrix} \begin{bmatrix} x \\ i \end{bmatrix} \begin{bmatrix} i \\ t \end{bmatrix}.
\end{aligned}$$

In the last line, we used Proposition A.3 (1) (a). Therefore the desired result follows. Similarly, we can prove Proposition A.3 (4) (b). □

Note that Proposition A.3 (4) implies a $q$-analogue of the identity (4.26) in Delsarte [6].

**Lemma A.4.**

$$\sum_{k=0}^{m} \begin{bmatrix} m-k \\ t \end{bmatrix} E_k(n,m;q;x) = q^{(m-t)x} \begin{bmatrix} m-x \\ m-t \end{bmatrix} \begin{bmatrix} n-t-x \\ m-t \end{bmatrix}$$

*Proof.* By the definition of the generalized Eberlein polynomial, the left hand side is written in

$$\sum_{k=0}^{m} \begin{bmatrix} m-k \\ t \end{bmatrix} \sum_{j=0}^{k} (-1)^{k-j} q^{jx+\binom{k-j}{2}} \begin{bmatrix} m-j \\ m-k \end{bmatrix} \begin{bmatrix} m-x \\ j \end{bmatrix} \begin{bmatrix} n-m+j-x \\ j \end{bmatrix}. \tag{A.2}$$

Interchanging the first summation and the second summation of (A.2), and using Proposition A.3 (2), it follows that (A.2) is equal to

$$\begin{aligned}
&\sum_{j=0}^{m} q^{jx} \begin{bmatrix} m-x \\ j \end{bmatrix} \begin{bmatrix} n-m+j-x \\ j \end{bmatrix} \sum_{k=j}^{m} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} m-k \\ t \end{bmatrix} \begin{bmatrix} m-j \\ m-k \end{bmatrix} \\
&\qquad = q^{(m-t)x} \begin{bmatrix} m-x \\ m-t \end{bmatrix} \begin{bmatrix} n-t-x \\ m-t \end{bmatrix}.
\end{aligned}$$

Therefore the desired result follows. □

Note that Lemma A.4 implies a $q$-analogue of the identity (36) in Tarnanen-Aaltonen-Goethals [12].

**Lemma A.5.**

$$\sum_{j=0\vee(v-r)}^{r\wedge(u-v)} q^{j(v-r+j)} \begin{bmatrix} u-v \\ j \end{bmatrix} \begin{bmatrix} v \\ r-j \end{bmatrix} K_s(r,l;q;j) = \begin{bmatrix} u-s \\ r-s \end{bmatrix} K_s(u,l;q;u-v) \tag{A.3}$$

*Proof.* By the definition of the generalized Krawtchouk polynomial and Proposition A.3 (1) (a), the left hand side of (A.3) is written in

$$\sum_{t=0}^{s}(-1)^{s-t}q^{tl+\binom{s-t}{2}}\begin{bmatrix}r-t\\r-s\end{bmatrix}\sum_{j=0\vee(v-r)}^{r\wedge(u-v)}q^{j(v-r+j)}\begin{bmatrix}u-v\\j\end{bmatrix}\begin{bmatrix}v\\t\end{bmatrix}\begin{bmatrix}v-t\\r-j-t\end{bmatrix}.$$

By Proposition A.3 (3) (b), we obtain

$$\sum_{j=0\vee(v-r)}^{r\wedge(u-v)}q^{j(v-r+j)}\begin{bmatrix}u-v\\j\end{bmatrix}\begin{bmatrix}v-t\\r-j-t\end{bmatrix}=\begin{bmatrix}u-t\\r-t\end{bmatrix}.$$

Using Proposition A.3 (1) (b), the left hand side of (A.3) is equal to

$$\begin{bmatrix}u-s\\r-s\end{bmatrix}\sum_{t=0}^{s}(-1)^{s-t}q^{tl+\binom{s-t}{2}}\begin{bmatrix}u-t\\u-s\end{bmatrix}\begin{bmatrix}u-(u-v)\\t\end{bmatrix}.$$

Therefore the desired result follows. □

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